Analytical treatment of critical collapse in 2+1 dimensional AdS spacetime: a toy model

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Abstract

We present an exact collapsing solution to 2+1 gravity with a negative cosmological constant minimally coupled to a massless scalar field, which exhibits physical properties making it a candidate critical solution. We discuss its global causal structure and its symmetries in relation with those of the corresponding continously self-similar solution derived in the $\Lambda = 0$ case. Linear perturbations on this background lead to approximate black hole solutions. The critical exponent is found to be $\gamma = 2/5$.

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1 Introduction

Since its discovery, the BTZ black hole solution [1] of 2+1 dimensional AdS gravity has attracted much interest because it represents a simplified context in which to study the classical and quantum properties of black holes. A line of approach which has been opened only recently [2, 3, 4, 5] concerns black hole formation through collapse of matter configurations coupled to 2+1 gravity with a negative cosmological constant. As first discovered in four dimensions by Choptuik [6], collapsing configurations which lie at the threshold of black hole formation exhibit properties, such as universality, power-law scaling of the black hole mass, and continuous or discrete self-similarity, which are characteristic of critical phenomena [7]. In the case of a spherically symmetric massless, minimally coupled scalar field, a class of analytical continuously self-similar (CSS) solutions was first given by Roberts [8, 9, 10]. These include critical solutions, lying at the threshold between black holes and naked singularities, and characterized by the presence of null central singularities. Linear perturbations of these solutions [11, 12] lead to approximate black hole solutions with a spacelike central singularity.

Numerical simulations of circularly symmetric scalar field collapse in 2+1 dimensional AdS spacetime were recently performed by Pretorius and Choptuik [2] and Husain and Olivier [3]. Both groups observed critical collapse, which was determined in [2] to be continuously self-similar near $r = 0$. In [4], Garfinkle has found a one-parameter family of exact CSS solutions of 2+1 gravity without cosmological constant, and argued that one of these solutions should give the behaviour of the full critical solution ($\Lambda \neq 0$) near the singularity.

The purpose of this paper is to present a new CSS solution to the field equations with $\Lambda = 0$ which can be extended to a threshold solution of the full $\Lambda \neq 0$ equations. The new $\Lambda = 0$ solution is derived in Sect. 3. It presents a null central singularity and, besides being CSS, possesses four Killing vectors. In Sect. 4 we address the extension of this CSS solution to a quasi-CSS solution of the full $\Lambda < 0$ problem, and show that the requirement of maximal symmetry selects a unique extension. This inherits the null central singularity of the $\Lambda = 0$ solution, and has the correct AdS boundary at spatial infinity. Finally, we perform in Sect. 5 the linear perturbation analysis in this background, find that it does lead to black hole formation, and determine the critical exponent.

2 CSS solutions

The Einstein equations for cosmological gravity coupled to a massless scalar field in (2+1) dimensions are

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{2.1}$$

with the stress-energy tensor for the scalar field

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial^\lambda \phi \partial_\lambda \phi. \tag{2.2}$$

The signature of the metric is (+ - -), and the cosmological constant Λ is negative for AdS spacetime, $\Lambda = -l^{-2}$. Static solutions of these equations include the BTZ black hole solutions [1] with a vanishing scalar field $\phi = 0$, and singular solutions when a non-trivial scalar field is coupled with the positive sign for the gravitational constant κ [13].

We shall use for radial collapse the convenient parametrisation of the rotationally symmetric line element in terms of null coordinates (u, v):

$$ds^2 = e^{2\sigma} du dv - r^2 d\theta^2, \tag{2.3}$$

with metric functions $\sigma(u, v)$ and $r(u, v)$. The corresponding Einstein equations and scalar field equation are

$$r_{,uv} = \tfrac{\Lambda}{2} r e^{2\sigma}, \tag{2.4}$$

$$2\sigma_{,uv} = \tfrac{\Lambda}{2} e^{2\sigma} - \kappa \phi_{,u} \phi_{,v}, \tag{2.5}$$

$$2\sigma_{,u} r_{,u} - r_{,uu} = \kappa r \phi_{,u}^2, \tag{2.6}$$

$$2\sigma_{,v} r_{,v} - r_{,vv} = \kappa r \phi_{,v}^2, \tag{2.7}$$

$$2r\phi_{,uv} + r_{,u}\phi_{,v} + r_{,v}\phi_{,u} = 0 \quad . \tag{2.8}$$

From the Einstein equations, the Ricci scalar is

$$R = -6\Lambda + 4\kappa e^{-2\sigma} \phi_{,u} \phi_{,v}. \tag{2.9}$$

It follows from (2.9) and (2.5) that the behavior of the solutions near the singularity is governed by the equations (2.4)-(2.8) with vanishing cosmological constant $\Lambda = 0$ (see also [5]). Assuming $\Lambda = 0$, Garfinkle has found [4] the following family of exact CSS solutions to these equations

$$\begin{aligned} ds^2 &= -A \left(\frac{(\sqrt{v} + \sqrt{-u})^4}{-uv} \right)^{\kappa c^2} du\, dv - \frac{1}{4}(v + u)^2 d\theta^2, \\ \phi &= -2c \ln(\sqrt{v} + \sqrt{-u}), \end{aligned} \tag{2.10}$$

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depending on an arbitrary constant c and a scale $A > 0$. In (2.10), u is retarded time, and $-v$ is advanced time. These solutions are continuously self-similar with homothetic vector $(u\partial_u + v\partial_v)$. An equivalent form of these CSS solutions, obtained by making the transformation

$$- u = (-\bar{u})^{2q}, \quad v = (\bar{v})^{2q} \qquad (1/2q = 1 - \kappa c^2) \qquad (2.11)$$

to the barred null coordinates $(\bar{u}, \bar{v})$, is

$$\begin{aligned} ds^2 &= -\bar{A}(\bar{v}^q + (-\bar{u})^q)^{2(2q-1)/q} \, d\bar{u} \, d\bar{v} - \frac{1}{4}(\bar{v}^{2q} - (-\bar{u})^{2q})^2 \, d\theta^2 \,, \\ \phi &= -2c\ln(\bar{v}^q + (-\bar{u})^q) \,. \end{aligned} \qquad (2.12)$$

The corresponding Ricci scalar is

$$R = \frac{4\kappa c^2}{A}(\bar{v}^q + (-\bar{u})^q)^{2(1-3q)/q}(-\bar{u})^{q-1}(\bar{v})^{q-1} \,. \qquad (2.13)$$

Garfinkle suggested that the line element (2.10) describes critical collapse in the sector $r = -(u + v)/2 \geq 0$, near the future point singularity $r = 0$ (where the Ricci scalar behaves, for $v \propto u$, as u^{-2}). The corresponding Penrose diagram (Fig. 1) is a triangle bounded by past null infinity $u \to -\infty$, the other null side $v = 0$, and the central regular timelike line $r = 0$. For $\kappa c^2 \geq 1$ ($q < 0$), the Ricci scalar

$$R \sim (\bar{v})^{q-1} \sim (v)^{(q-1)/2q} \qquad (2.14)$$

is regular near $v = 0$, which moreover turns out to be at infinite geodesic distance. To show this, we consider the geodesic equation

$$(e^{2\sigma}\dot{v})\dot{} = -2rr_{,u}\dot{\theta}^2 = -2l^2r^{-3}r_{,u} \qquad (2.15)$$

(l constant) near $v = 0$, u constant, which gives $v \propto (ls)^{4q}$ for $l \neq 0$, or s^{2q} for $l = 0$, so that in all cases the affine parameter $s \to \infty$ for $v \to 0$, and the spacetime is geodesically complete. For $\kappa c^2 < 1$ ($q > 0$), we see from (2.13) that the null line $v = 0$ is a curvature singularity if $\kappa c^2 < 1/2$ ($q < 1$). If $1/2 \leq \kappa c^2 < 1$ ($q \geq 1$), the surface $v = 0$ is regular. However, as discussed by Garfinkle, the metric (2.12) can be extended through this surface only for $q = n$, where n is a positive integer. For n even, the extended spacetime is made of two symmetrical triangles joined along the null side $\bar{v} = 0$, and has two coordinate singularities $r = 0$, one timelike ($\bar{u} - \bar{v} = 0$) and one spacelike ($\bar{u} + \bar{v} = 0$), but no curvature singularity. For n odd, one of the $r = 0$ sides becomes a future spacelike curvature singularity ($e^{2\sigma} = 0$), similar to that of

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Brady's supercritical solutions for scalar field collapse in (3+1) dimensions [9], except for the fact that in the present case the singularity is not hidden behind a spacelike apparent horizon (Fig. 2).

Let us point out that, besides the solutions (2.10), the system (2.4)-(2.8) also admits for $\Lambda = 0$ another family of CSS solutions

$$
\begin{aligned}
ds^2 &= A\left(\frac{(\sqrt{v} - \sqrt{-u})^4}{-uv}\right)^{\kappa c^2} du\, dv - \frac{1}{4}(v+u)^2\, d\theta^2\,, \\
&= \bar{A}(\bar{v}^q - (-\bar{u})^q)^{2(2q-1)/q}\, d\bar{u}\, d\bar{v} - \frac{1}{4}(\bar{v}^{2q} - (-\bar{u})^{2q})^2\, d\theta^2\,, \quad (2.16)
\end{aligned}
$$

with $\phi = -2c\ln(\sqrt{v} - \sqrt{-u})$, and we choose $A > 0$ and consider the sector $0 \le v \le -u$. These solutions have a future spacelike central $(r = 0)$ curvature singularity at $(-\bar{u})^q = \bar{v}^q$ (where the Ricci scalar (2.13) diverges) for all $q < 0$ or $q > 0$ (implying $q > 1/2$). For $q < 0$, the Penrose diagram is a triangle bounded by past null infinities $\bar{u} \to -\infty$ and $\bar{v} = 0$ (which is at infinite geodesic distance). For $q > 0$, geodesics terminate at $\bar{v} = 0$, unless $q = n$ integer. For n even, the extended spacetime has two central curvature singularities $r = 0$, one spacelike and the other timelike. The extended spacetime for n odd is more realistic. In this case the extension from $\bar{v} > 0$ to $\bar{v} < 0$ amounts to replacing (2.16) with $A > 0$ by the original Garfinkle solution (2.10) with $A > 0$, the resulting Penrose diagram being that of Fig. 2.

3 A new CSS solution for $\Lambda = 0$

Among the one-parameter (c or q) family of CSS solutions (2.10), the special solution, corresponding to $\kappa c^2 = 1$,

$$
ds^2 = A(\sqrt{v} + \sqrt{-u})^4 \frac{du}{u}\frac{dv}{v} - \frac{1}{4}(v+u)^2\, d\theta^2\,, \quad\quad (3.1)
$$

is singled out by the fact that the transformation (2.11) breaks down for this value. The transformation appropriate to this case,

$$
-u = 2e^{-U}\,, \quad v = 2e^V = 2e^{U-2T} \quad\quad (3.2)
$$

(with $T \ge U$ for $u + v \le 0$) transforms the solution (3.1) to

$$
\begin{aligned}
ds^2 &= e^{-2U}[-4A(1 + e^{U-T})^4\, dU\, dV - (1 - e^{2(U-T)})^2\, d\theta^2]\,, \\
\phi &= U - 2\ln(1 + e^{U-T}) \quad\quad (3.3)
\end{aligned}
$$

(we use from now on units such that $\kappa = 1$, and have dropped an irrelevant additive constant from ϕ).

Starting from this special CSS solution of the Garfinkle class, we now derive, by a limiting process, a new CSS solution which, as we shall see, exhibits a null singularity. We translate T to $T - T_0$, and take the late-time limit $T_0 \to -\infty$, leading to the new CSS solution (written for $A = -1/2$)

$$ds^2 = e^{-2U}(2dU\,dV - d\theta^2), \quad \phi = U, \tag{3.4}$$

with a very simple form which is reminiscent of the Hayward critical solution for scalar field collapse in 3+1 dimensions [12],

$$ds^2 = e^{2\rho}(2d\tau^2 - 2d\rho^2 - d\Omega^2), \quad \phi = \tau. \tag{3.5}$$

The transformation

$$\bar{u} = -e^{-2U}, \quad \bar{v} = V \tag{3.6}$$

leads from (3.4) to the even more simple form of this solution

$$ds^2 = d\bar{u}\,d\bar{v} + \bar{u}\,d\theta^2, \quad \phi = -\frac{1}{2}\ln(-\bar{u}), \tag{3.7}$$

which is reminiscent of the other form of the Hayward solution

$$ds^2 = 2\,d\bar{u}\,d\bar{v} + \bar{u}\bar{v}\,d\Omega^2, \quad \phi = -\frac{1}{2}\ln(-\bar{u}/\bar{v}). \tag{3.8}$$

The solution (3.4) or (3.7) is continuously self-similar, with homothetic vector

$$K = \partial_U = -2\bar{u}\partial_{\bar{u}}. \tag{3.9}$$

It also has a high degree of symmetry, with 4 Killing vectors

$$\begin{aligned}
L_1 &= \partial_U + 2V\partial_V + \theta\partial_\theta, \\
L_2 &= \theta\partial_V + U\partial_\theta, \\
L_3 &= \partial_V, \\
L_4 &= \partial_\theta,
\end{aligned} \tag{3.10}$$

generating the solvable Lie algebra

$$\begin{aligned}
[L_1, L_2] &= L_4 - L_2, & [L_2, L_3] &= 0, \\
[L_1, L_3] &= -2L_3, & [L_2, L_4] &= -L_3, \\
[L_1, L_4] &= -L_4, & [L_3, L_4] &= 0.
\end{aligned} \tag{3.11}$$

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The Ricci scalar (2.9) is identically zero for the solution (3.4), for which the sole nonvanishing Ricci tensor component is $R_{UU} = 1$. It follows that this metric is devoid of curvature singularity. However there is an obvious coordinate singularity at $U \to +\infty$, or $\bar{u} = 0$ (where $r = 0$). To determine the nature of this singularity, we study geodesic motion in the spacetime (3.7). The geodesic equations are integrated by

$$\dot{\bar{u}} = \pi \,, \quad \bar{u}\dot{\theta} = l \,, \quad \pi\dot{\bar{v}} + l\dot{\theta} = \varepsilon \,, \tag{3.12}$$

where π and l are the constants of the motion associated with the Killing vectors L_3 and L_4, and the sign of ε depends on that of ds^2 along the geodesic. The null line $\bar{u} = 0$ can be reached only by those geodesics with $\pi \neq 0$. Then, the third equation (3.12) integrates to

$$\bar{v} = \frac{\varepsilon}{\pi^2}\bar{u} - \frac{l}{\pi}\theta + const. = \frac{\varepsilon}{\pi^2}\bar{u} - \frac{l^2}{\pi^2}\ln(-\bar{u}) + const.. \tag{3.13}$$

It follows that nonradial geodesics ($l \neq 0$) terminate at $\bar{u} = 0, \bar{v} \to +\infty$, while radial geodesics ($l = 0$), which behave as in cylindrical Minkowski space, can be continued through the null line $\bar{u} = 0$ to $\bar{u} \to +\infty$. So in this sense only the endpoint $\bar{v} \to +\infty$ of the null line $\bar{u} = 0$ is singular. However formal analytic continuation of the metric (3.7) from $\bar{u} < 0$ to $\bar{u} > 0$ involves a change of signature from (+ - -) to (+ - +), leading to the appearance of closed timelike curves. So the null line $\bar{u} = 0$ corresponds to a singularity in the causal structure of the spacetime, analogous to the central singularity in the causal structure of the BTZ black holes [1]. The resulting Penrose diagram, reminiscent of that of the Hayward critical solution [12], is a diamond bound by three lines at null infinity ($\bar{v} = -\infty, \bar{u} = -\infty, \bar{v} = +\infty$) and the null singularity $\bar{u} = 0$ (Fig. 3).

4   Extending the new solution to $\Lambda \neq 0$

In the preceding section we have found an exact solution for scalar field collapse with $\Lambda = 0$, which presents a central null singularity. This property makes it a candidate threshold solution, lying at the boundary between naked singularities and black holes. However black holes exist only for $\Lambda < 0$, so the solution (3.7) can only represent the behavior of the true threshold solution near the central singularity, where the cosmological constant can be neglected. This hypothetical $\Lambda < 0$ solution cannot be self-similar, essentially because the scale is fixed preferentially by the cosmological constant

[2]. So what we need is to find some other way to extend (3.7) to a solution of the full system of Einstein equations with $\Lambda < 0$.

A first possible approach is to expand this solution in powers of Λ, with the zeroth order given by the CSS solution (3.7). In the parametrisation (2.3), this zeroth order is (dropping the bars in (3.7))

$$r_0 = (-u)^{1/2}, \quad \sigma_0 = 0, \quad \phi_0 = -\frac{1}{2}\ln|u|. \tag{4.1}$$

We look for an approximate solution to first order in Λ of the form

$$r = (-u)^{1/2} + \Lambda r_1, \quad \sigma = \Lambda \sigma_1, \quad \phi = -\frac{1}{2}\ln|u| + \Lambda \phi_1, \tag{4.2}$$

with the boundary condition that the fonctions r_1, σ_1 and ϕ_1 vanish on the central singularity $u = 0$. Eq (2.4) gives

$$r_1 = (-u)^{1/2}(\frac{1}{3}uv + f(u)), \tag{4.3}$$

with $f(0) = 0$. Then, the linearized Eq. (2.7) gives

$$2r_0^{1/2}(r_0^{1/2}\phi_{1,v})_{,u} = -r_{1,v}\phi_{0,u} = \frac{1}{6}(-u)^{1/2}, \tag{4.4}$$

which is solved by

$$\phi_1 = (\frac{1}{15}uv + g(u)). \tag{4.5}$$

The linearized Eq. (2.5)

$$2\sigma_{1,uv} = 1 - \phi_{0,u}\phi_{1,v} = \frac{8}{15} \tag{4.6}$$

then gives

$$\sigma_1 = \frac{4}{15}uv + h(u). \tag{4.7}$$

Finally Eq. (2.5) leads to the relation between the arbitrary functions f, g, h

$$uf''(u) + f'(u) = g'(u) + h'(u). \tag{4.8}$$

Not only does this first order solution break the continuous self-similarity generated by (3.9), as expected, but it also breaks the isometry group generated by the Killings (3.10) down to $U(1)$ (generated by $L_4 = \partial_\theta$), except in the special case $f = g = h = 0$, where the Killing subalgebra (L_1, L_4)

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remains. This suggests looking for an exact $\Lambda < 0$ extension of the $\Lambda = 0$ CSS solution of the form

$$ds^2 = e^{2\sigma(x)}dudv + u\rho^2(x)d\theta^2, \quad \phi = -\frac{1}{2}\ln|u| + \psi(x), \qquad (4.9)$$

with $x = uv$. This will automatically preserve to all orders the Killing subalgebra (L_1, L_4). Inserting this ansatz into the field equations (2.4)-(2.8) leads to the system

$$x\rho'' + \frac{3}{2}\rho' = \frac{\Lambda}{2}\rho e^{2\sigma}, \quad (4.10)$$

$$2(x\sigma'' + \sigma') + \psi'(x\psi' - \frac{1}{2}) = \frac{\Lambda}{2}e^{2\sigma}, \quad (4.11)$$

$$x^2(-\rho'' + 2\rho'\sigma' - \rho\psi'^2) + x(-\rho' + \rho(\sigma' + \psi')) = 0 \quad (4.12)$$

$$-\rho'' + 2\rho'\sigma' - \rho\psi'^2 = 0 \quad (4.13)$$

$$2x(\rho\psi')' + \frac{5}{2}\rho\psi' = \frac{1}{2}\rho'. \quad (4.14)$$

($' = d/dx$). The unique, maximally symmetric extension of the CSS solution (3.7) reducing to (3.7) near $u = 0$ is the solution of the system (4.10)-(4.14) with the boundary conditions

$$\rho(0) = 1, \quad \sigma(0) = 0, \quad \psi(0) = 0. \qquad (4.15)$$

The comparison of (4.12) and (4.13) yields

$$\rho = e^{\sigma + \psi}. \qquad (4.16)$$

The combination (4.10) + x(4.13) then gives, together with (4.16),

$$x(2\sigma'^2 + 2\sigma'\psi' - \psi'^2) + \frac{3}{2}(\sigma' + \psi') = \frac{\Lambda}{2}e^{2\sigma}. \qquad (4.17)$$

The third independent equation is for instance (4.11):

$$2(x\sigma'' + \sigma') + \psi'(x\psi' - \frac{1}{2}) = \frac{\Lambda}{2}e^{2\sigma}. \qquad (4.18)$$

Using these last two equations with the boundary conditions (4.15), one can in principle write down series expansions for $\sigma(x)$ and $\psi(x)$. Another simple relation, deriving from (4.13) and (4.16), is

$$\sigma'' + \psi'' - \sigma'^2 + 2\psi'^2 = 0. \qquad (4.19)$$

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We are interested in the behavior of this extended solution in the sector $u < 0$, $v > 0$, i.e. $x < 0$. In this sector, Eqs. (4.10), (4.14) and (4.11) can be integrated to

$$(-x)^{3/2}\rho' = \frac{\Lambda}{2}\int_x^0 (-x)^{1/2}\rho e^{2\sigma}\,dx, \tag{4.20}$$

$$(-x)^{5/4}\rho\psi' = \frac{1}{4}\int_x^0 (-x)^{1/4}\rho'\,dx, \tag{4.21}$$

$$-x\sigma' = \frac{1}{2}\int_x^0 (\frac{\Lambda}{2}e^{2\sigma} + \psi'(\frac{1}{2} - x\psi'))dx. \tag{4.22}$$

As long as $\rho > 0$, Eq. (4.20) (with $x < 0$, $\Lambda < 0$) implies $\rho' < 0$, so that $\rho(x)$ decreases to 1 when x increases to 0. It then follows from (4.21) that $\psi' < 0$. Also, (4.21) can be integrated by parts to

$$x\psi' = \frac{1}{4} - \frac{1}{16(-x)^{1/4}\rho}\int_x^0 (-x)^{-3/4}\rho\,dx, \tag{4.23}$$

showing that $x\psi' < 1/4$. It then follows from (4.22) that $\sigma' < 0$. So, as x decreases, the functions ρ and $e^{2\sigma}$ increase and possibly go to infinity for a finite value $x = x_1$. If this is the case, the behavior of these functions near x_1 must be

$$\rho = \rho_1(\frac{1}{\bar{x}} + \frac{1}{4x_1} - \frac{\bar{x}\ln(\bar{x})}{48x_1^2} + ...)$$

$$e^{2\sigma} = \frac{4x_1}{\Lambda\bar{x}^2}(1 + \frac{\bar{x}^2\ln(\bar{x})}{48x_1^2} + ...)$$

$$\psi = \psi_1 + \frac{\bar{x}}{4x_1} - \frac{\bar{x}^2}{32x_1^2}\ln(\bar{x}) + ... \tag{4.24}$$

$(\bar{x} = x - x_1)$.

These expectations are borne out by the actual numerical solution of the system

$$x\rho'' + \frac{3}{2}\rho' = -\rho e^{2\sigma},$$

$$-\rho''\rho + 4\rho\rho'\sigma' = \rho'^2 + \rho^2\sigma'^2, \tag{4.25}$$

(this last equation comes from (4.13) where ψ' is given by derivation of (4.16)) where we have set $\Lambda = -2$, with the boundary counditions $\rho(0) = 1$, $\rho'(0) = -2/3$ (see eqs. (4.3) and (4.2)), $\sigma(0) = 0$. The plots of the functions

$\rho(x)$, $\sigma(x)$ and $\psi'(x)$ are given in Figs. (4,5,6,). The value of x_1 is found to be approximately -1.94 (i.e. $\Lambda x_1 = +3.88$).

The coordinate transformation[1]

$$u = \Lambda^{-1} e^{-\bar{U}}, \quad v = e^{\bar{V}} \qquad (\bar{U} = \bar{T} - \bar{R}, \quad \bar{V} = \bar{T} + \bar{R}) \qquad (4.26)$$

leads to $x = \Lambda^{-1} e^{2\bar{R}}$ and, on account of (4.9) and (4.16), to the form of the metric

$$ds^2 = -\Lambda^{-1} e^{2(\sigma(\bar{R}) + \bar{R})} (d\bar{U} d\bar{V} - e^{2\psi(\bar{R}) - \bar{V}} d\theta^2). \qquad (4.27)$$

Near the spacelike boundary $\bar{R} = \bar{R}_1$ of the spacetime, the collapsing metric and scalar field behave, from (4.24), as

$$ds^2 \simeq -\Lambda^{-1} (\bar{R}_1 - \bar{R})^{-2} (d\bar{T}^2 - d\bar{R}^2 - e^{\bar{T}_1 - \bar{T}} d\theta^2), \quad \phi = \phi_1 + \bar{T}/2 \quad (4.28)$$

$(\bar{R} - \bar{R} \simeq \bar{x}/2x_1)$. This metric is asymptotically AdS, as may be shown by making the further coordinate transformation,

$$\bar{R} - \bar{R}_1 = -2/XT, \quad \bar{T} - \bar{T}_1 = 2\ln(T/2), \qquad (4.29)$$

leading to

$$ds^2 \simeq -\Lambda^{-1} \left(X^2 dT^2 - \frac{dX^2}{X^2} - X^2 d\theta^2 \right), \quad \phi = \phi_1 + \ln(T/2). \qquad (4.30)$$

The next-to-leading terms in the metric containing logarithms, this asymptotic behavior differs from that of BTZ black holes.

It follows from this discussion that the Penrose diagram of the $\Lambda < 0$ threshold solution in the sector $v > 0$, $u < 0$ is a triangle bounded by the null line $v = 0$, the null causal singularity $u = 0$, and the spacelike AdS boundary $X \to \infty$. The null singularity $u = 0$ remains naked, i.e. is not hidden behind a trapping horizon, which would correspond to

$$\partial_v r = -(-u)^{3/2} \rho'(x) = 0, \qquad (4.31)$$

because $\rho' < 0$ (as discussed above) implies that the only solution of this equation is $u = 0$.

For the sake of completeness, let us also discuss the behavior of the solution of the system (4.10)-(4.14) in the sector $x > 0$. In this case, one can write down integro-differential equations similar to (4.20)-(4.22), from

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[1]We have taken care that in (4.9) u has the dimension of a length squared while v is dimensionless.

which one again derives that $\rho' < 0$, $\psi' < 0$ and $\sigma' < 0$. It follows that the metric function $e^{2\sigma}$ decreases as x increases, eventually vanishing for a finite value $x = x_0$, corresponding to a spacelike curvature singularity (this has been confirmed numerically). The behavior of the solution near this singularity is found to be

$$\psi \simeq \gamma \ln(x_0 - x), \quad \sigma \simeq \frac{\gamma^2}{2} \ln(x_0 - x), \quad \rho \propto (x_0 - x) \qquad (\gamma = \sqrt{3} - 1),$$
(4.32)

and the coordinate transformation $u = e^U, v = e^V (x = e^{2T})$ leads to the form of the metric near the singularity

$$ds^2 \simeq (T_0 - T)^{\gamma^2}(dT^2 - dR^2) + e^{R_0 - R}(T_0 - T)^2 d\theta^2.$$
(4.33)

5   Perturbations

To check whether the quasi-CSS solution (4.9) of the full $\Lambda \neq 0$ problem determined in the preceding section is indeed a threshold solution, we now study linear perturbations of this solution. Our treatment will follow the analysis of perturbations of critical solutions in the case of scalar field collapse in 3+1 dimensions [11, 12].

The relevant time parameter in critical collapse being the retarded time $U = -(1/2)\ln(-u)$ (the "scaling variable" of [11]), we expand these perturbations in modes proportional to $e^{kU} = (-u)^{-k/2}$, with k a complex constant. We recall that only the modes with $Re\, k > 0$ grow as $U \to +\infty$ ($u \to -0$) and lead to black hole formation, whereas those with $Re\, k < 0$ decay and are irrelevant. The other relevant variable is the "spatial" coordinate $x = uv$, and the perturbations are decomposed as

$$\begin{aligned}
r &= (-u)^{1/2}(\rho(x) + (-u)^{-k/2}\tilde{r}(x)), \\
\phi &= -\frac{1}{2}\ln|u| + \psi(x) + (-u)^{-k/2}\tilde{\phi}(x), \\
\sigma &= \sigma(x) + (-u)^{-k/2}\tilde{\sigma}(x).
\end{aligned}$$
(5.1)

Then, the Einstein equations (2.4)-(2.8) are linearized in $\tilde{r}$, $\tilde{\phi}$, $\tilde{\sigma}$, using

$$\delta\phi_{,u} = -(-u)^{-k/2-1}(x\tilde{\phi}' - \frac{k}{2}\tilde{\phi}), \quad \delta\phi_{,v} = -(-u)^{-k/2+1}\tilde{\phi}'.$$
(5.2)

The resulting equations are homogeneous in u, which drops out, and the linearized system reduces to

$$x\tilde{r}'' + (-k/2 + 3/2)\tilde{r}' = \frac{\Lambda}{2}e^{2\sigma}(\tilde{r} + 2\rho\tilde{\sigma}),$$
(5.3)

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$$2x\tilde{\sigma}'' + (-k+2)\tilde{\sigma}' = \Lambda e^{2\sigma}\tilde{\sigma} - (2x\psi' - 1/2)\tilde{\phi}' + (k/2)\psi'\tilde{\phi}, \tag{5.4}$$

$$-(-k+1)x\tilde{r}' + ((-k+1)x\sigma' - (k^2-1)/4)\tilde{r} + \rho x\tilde{\sigma}' - k(x\rho' + \rho/2)\tilde{\sigma} =$$
$$-\rho(x\tilde{\phi}' - k(1/2 - x\psi')\tilde{\phi}) + (1/4 - x\psi')\tilde{r}, \tag{5.5}$$

$$2(\rho'\tilde{\sigma}' + \sigma'\tilde{r}') - \tilde{r}'' = \psi'(2\rho\tilde{\phi}' + \psi'\tilde{r}), \tag{5.6}$$

$$2x\rho\tilde{\phi}'' + (2x\rho' + (-k+5/2)\rho)\tilde{\phi}' - (k/2)\rho'\tilde{\phi} + (2x\psi' - 1/2)\tilde{r}'$$
$$+(2x\psi'' + (-k/2+5/2)\psi')\tilde{r} = 0. \tag{5.7}$$

What is the number of the independent constants for this system? The perturbed Klein-Gordon equation (5.7) is clearly redundant, while Eqs. (5.5) and (5.6) are constraints. So, as in the (3+1)-dimensional case [11, 12], the order of the system is four, and the general solution depends on four integration constants. However, one of these four independent solutions corresponds to a gauge mode and is irrelevant. The parametrisation (4.9) is invariant under infinitesimal coordinate transformations $v \to v + f(v)$. For $f(v) = -\alpha v^{1+k/2}$, these lead to $x \to x - \alpha(-u)^{-k/2}(-x)^{1+k/2}$, giving rise to the gauge mode

$$
\begin{aligned}
\tilde{r}_k(x) &= \alpha(-x)^{1+k/2}\rho'(x), \\
\tilde{\phi}_k(x) &= \alpha(-x)^{1+k/2}\psi'(x), \\
\tilde{\sigma}_k(x) &= \alpha[(-x)^{1+k/2}\sigma'(x) - \frac{k+2}{4}(-x)^{k/2}],
\end{aligned}
\tag{5.8}
$$

which solves identically the system (5.3)-(5.7). So, up to gauge transformations, the general solution of this system depends only on three independent constants.

These will be determined, together with the possible values of k (the eigenfrequencies) by enforcing appropriate and reasonable boundary conditions. We shall use here the "weak boundary conditions" of [12] on the boundaries $u = 0$ and $x = x_1$ ($X \to \infty$)

$$\lim_{u \to 0} r^{-1} \neq 0, \quad \lim_{x \to x_1} r \neq 0, \tag{5.9}$$

together with the condition

$$\tilde{r}(0) = 0, \tag{5.10}$$

which guarantees that the singularity of the perturbed solution starts smoothly from that of the unperturbed one. On the third boundary $v = 0$, we shall impose a stronger condition by requiring that the perturbations are analytic in v, in order for the perturbed solution to be extendible beyond $v = 0$ to negative values of v at finite u.

First, we consider the region $x \to 0$ where, according to Eqs. (4.1), (4.3), (4.5) and (4.7),

$$\rho \simeq 1 + \frac{1}{3}\Lambda x, \quad e^{2\sigma} \simeq 1 + \frac{4}{15}\Lambda x, \quad \psi \simeq \frac{1}{15}\Lambda x. \tag{5.11}$$

Let us assume a power-law behavior

$$\tilde{r}(x) \sim a(-x)^p \tag{5.12}$$

where p is a constant to be determined. Then Eqs. (5.3), (5.4) and (5.6) can be approximated near $x = 0$ as

$$x\tilde{r}'' + (-k/2 + 3/2)\tilde{r}' \simeq \Lambda\tilde{\sigma}, \tag{5.13}$$

$$x\tilde{\sigma}'' + (-k/2 + 1)\tilde{\sigma}' \simeq \frac{1}{4}\tilde{\phi}' \tag{5.14}$$

$$2\rho'\tilde{\sigma}' - \tilde{r}'' \simeq 2\rho\psi'\tilde{\phi}'. \tag{5.15}$$

Eliminating the functions $\tilde{\sigma}$ and $\tilde{\phi}$ between these three equations and using Eq. (5.11), we obtain the fourth-order equation

$$4x^2\tilde{r}'''' + (-4k + 13)x\tilde{r}''' + (k/2 - 1)(2k - 5)\tilde{r}'' \simeq 0, \tag{5.16}$$

which implies the power-law behavior (5.12) with the exponent p constrained by

$$p(p - 1)(p - k/2 - 3/4)(p - k/2 - 1) = 0. \tag{5.17}$$

Obviously the root $p = k/2 + 1$ corresponds to the gauge mode (5.8) and must be discarded as irrelevant. As a consequence the general solution near $x = 0$ can be given in terms of three independent constants as

$$\tilde{r}(x) \sim A + B(-x) + \Lambda C(-x)^{3/4+k/2}, \tag{5.18}$$

$$\tilde{\sigma}(x) \sim -\frac{A}{2} + \Lambda^{-1}\frac{(k-3)B}{2} - \frac{5C}{8}(k + \frac{3}{2})(-x)^{-1/4+k/2}, \tag{5.19}$$

$$\tilde{\phi}(x) \sim \frac{(1-k)A}{2} - \Lambda^{-1}\frac{(k-3)B}{2} + \frac{5C}{8}(k + \frac{3}{2})(-x)^{-1/4+k/2} \tag{5.20}$$

Let us note that this solution remains valid in the limit $\Lambda \to 0$, leading to the limiting solution $\tilde{r} \sim A + B(-x)$ (with $B = 0$ for $k \neq 3$), which could also be obtained directly by solving the equation $\tilde{r}'' = 0$ which results from (5.6) in the limit $\Lambda \to 0$, together with the stronger condition (from Eq. (5.3)) $(k - 3)\tilde{r}' = 0$.

Now we enforce the boundary conditions at $x = 0$. For $k > 0$, $\tilde{r}$ is dominated by its first constant term in (5.18), so that the condition (5.10) can only be satisfied for $u \to 0$ if

$$A = 0. \tag{5.21}$$

Then, for $k > 1/2$, $\tilde{r}$ is dominated by its second term $-Bx$, leading to a perturbation $(-u)^{1/2-k/2}\tilde{r}(x)$ which blows up as $u \to 0$ and violates (5.9) unless

$$k \leq 3. \tag{5.22}$$

Then we impose the condition of analyticity in v at fixed u. This is satisfied if

$$k = 2n - 3/2, \tag{5.23}$$

where n is a positive integer. Combining eqs. (5.22) and (5.23) we find that k has only two positive eigenvalues

$$k = 1/2, \quad k = 5/2. \tag{5.24}$$

However, in the above analysis we have disregarded the fact that $k = 1/2$ is a double root of the secular equation (5.17). For $k = 1/2$ the correct behavior of the general solution near $x = 0$ is

$$\tilde{r}(x) \sim A + B(-x) + \Lambda C(-x)\ln|x|, \tag{5.25}$$

$$\tilde{\sigma}(x) \sim -\frac{A}{2} - \Lambda^{-1}\frac{5B}{4} - \frac{9C}{4} - \frac{5C}{4}\ln|x| \tag{5.26}$$

$$\tilde{\phi}(x) \sim \frac{A}{4} + \Lambda^{-1}\frac{5B}{4} + \frac{9C}{4} + \frac{5C}{4}\ln|x|, \tag{5.27}$$

which satisfies the condition of analyticity only if $C = 0$.

At the AdS boundary ($x \to x_1$) the leading behaviour of the background is, from Eqs. (4.24),

$$\rho \simeq \frac{\rho_1}{x - x_1}, \quad e^{2\sigma} \simeq \left(\frac{4x_1}{\Lambda}\right)\frac{1}{(x - x_1)^2}, \quad \psi \simeq \psi_1. \tag{5.28}$$

We again assume a power-law behavior

$$\tilde{\sigma} \sim b\bar{x}^q \tag{5.29}$$

($\bar{x} = x - x_1$). Then Eq. (5.4), where $\tilde{\phi}$ can be neglected, gives

$$q(q - 1) = 2, \tag{5.30}$$

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i.e. $q = -1$ or $q = 2$. Then, Eq. (5.3) reduces near $\bar{x} = 0$ to

$$\tilde{r}'' - 2\bar{x}^{-2}\tilde{r} \simeq 4b\rho_1\bar{x}^{q-3}. \tag{5.31}$$

If $q = -1$, the behavior of the solution is governed by the right-hand side, i.e. $\tilde{r} \propto \bar{x}^{-2}$, which violates the boundary condition (5.9) for $x \to x_1$. So the behavior $\tilde{\sigma} \sim b\bar{x}^{-1}$ must be excluded, which fixes another integration constant $D = 0$ (where D is a linear combination of B and C). Then, the generic behavior of the solution of Eq. (5.31) with $q = 2$ is governed by that for the homogeneous equation, i.e.

$$\tilde{r} \sim \frac{E}{x - x_1}. \tag{5.32}$$

This is consistent with the boundary condition (5.9), and is an admissible small perturbation if its amplitude is small enough, $E \ll \rho_1$.

For $k = 1/2$, we have seen that two of the three integration constants in (5.25)-(5.27) are fixed ($A = C = 0$) by condition (5.10) and the analyticity condition, while the weak boundary condition at the AdS boundary fixes a third constant $D = 0$. However this is impossible, as the perturbation amplitude must remain as a free parameter. So the mode $k = 1/2$ cannot satisfy all our boundary conditions, and we are left with a single eigenmode,

$$k = 5/2, \tag{5.33}$$

completely determined up to an arbitrary amplitude by the two conditions $A = D = 0$.

The corresponding perturbed metric function r behaves near $x = 0$ as

$$r \simeq (-u)^{1/2}[1 + \frac{1}{3}\Lambda x - (-u)^{-5/4}Bx]. \tag{5.34}$$

For $B < 0$, the central singularity $r = 0$ is approximately given by

$$(-u)^{1/4} \simeq -Bv. \tag{5.35}$$

Our boundary conditions guarantee that it starts at $u = v = 0$ (as for the unperturbed solution) and then becomes spacelike in the region $v > 0$. This singularity is hidden behind a trapping horizon (defined by Eq. (4.31)) which, near $x = 0$, is null,

$$(-u)^{5/4} = \frac{3B}{\Lambda} \tag{5.36}$$

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(a null trapping horizon was also found in [12]). Let us point out the crucial role played by the cosmological constant Λ in the formation of this trapping horizon. For $\Lambda = 0$, $\rho(x) = 1$, while, as discussed after Eq. (5.20), the perturbation $\tilde{r}$ with the boundary condition (5.10) vanishes for $\Lambda = 0$, so that the perturbed radial function r is (as in [4]) identical to the CSS one, and the trapping horizon does not exist. Near the AdS boundary $x \to x_1$, it follows from (5.28) and (5.32) that both the central singularity and the trapping horizon are tangent to the null line

$$(-u)^{5/4} = -E(\frac{4x_1}{\Lambda})^{-1/2} . \tag{5.37}$$

Thus, perturbations of the quasi-CSS solution lead to black hole formation, showing that this solution is indeed a threshold solution, and is a candidate to describe critical collapse. Near-critical collapse is characterized by a critical exponent γ, defined by the scaling relation $Q \propto |p - p^*|^{s\gamma}$, for a quantity Q with dimension s depending on a parameter p (with $p = p^*$ for the critical solution). Choosing for Q the radius r_{AH} of the apparent horizon, and identifying $p - p^*$ with the perturbation amplitude B, we obtain from (5.36)

$$r_{AH} \simeq (\frac{3B}{\Lambda})^{2/5}, \tag{5.38}$$

leading to the value of the critical exponent $\gamma = 2/5$, in agreement with the renormalization group argument [14] leading to $\gamma = 1/k$.

6   Conclusion

We have discussed in detail the causal structure of the Garfinkle CSS solutions (2.10) to the $\Lambda = 0$ Einstein-scalar field equations. From a special solution of this class, we have derived by a limiting process a new CSS solution, which we have extended to a unique solution of the full $\Lambda < 0$ equations, describing collapse of the scalar field onto a null central singularity. This is not a curvature singularity (all the curvature invariants remain finite), but a singularity in the causal structure similar to that of the BTZ black hole. Finally, we have analyzed linear perturbations of the $\Lambda < 0$ solution, found a single eigenmode $k = 5/2$, checked that this mode does indeed give rise to black holes, and determined the critical exponent $\gamma = 2/5$.

For comparison, Choptuik and Pretorius [2] derived, by analysing the observed scaling behavior of the maximum scalar curvature, the value $1.15 < \gamma < 1.25$ for the critical exponent. This value is different from the value

$\gamma \sim 0.81$ obtained in the numerical analysis of Husain and Olivier [3] from the scaling behavior of the apparent horizon radius. Our value $\gamma = 0.4$, while significantly smaller than these two conflicting estimates, is of the order of the theoretical value $\gamma = 1/2$ derived either from the analysis of dust-ring collapse [15], of black hole formation from point particle collisions [16], or of the $J = 0$ to $J \neq 0$ transition of the BTZ black hole [17].

It is worth mentioning here that, even though they were obtained for a vanishing cosmological constant and thus solve the $\Lambda \neq 0$ equations only near the singularity, the Garfinkle CSS solutions are, for the particular value (chosen in order to better fit the numerical curves) $c = (7/8)^{1/2} \simeq 0.935$, in good agreement [4] with the numerical results of [2] at an intermediate time. The fact that this value is close to 1 suggests that the $c = 1$ CSS solution (3.3) approximately describes near-critical collapse at intermediate times. If this the case, then it would not be surprising if its late-time limit, our new CSS solution Eq. (3.4), gives a good description of exactly critical collapse near the singularity. A fuller understanding of the relationship between the numerically observed near-critical collapse and these various $\Lambda = 0$ CSS solutions could be achieved by extending them to $\Lambda < 0$, as done in the present work for the special solution (3.7).

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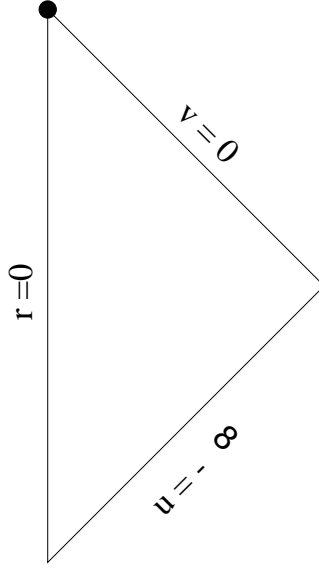[17] J.P. Krisch and E.N. Glass, "Critical exponents for Schwarzschild-Kerr and BTZ systems", gr-qc/0102075

Figure 1: Penrose diagram of the solutions eq. (2.12) for $q < 0$.
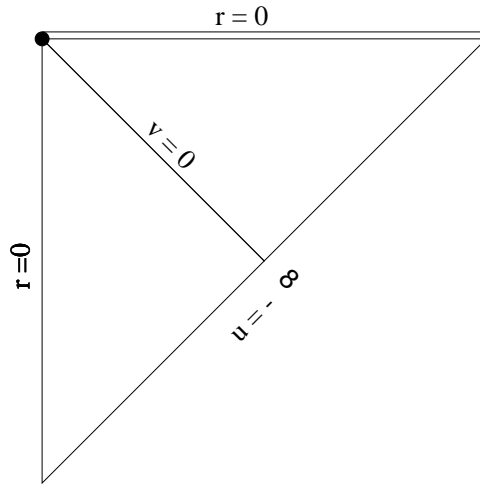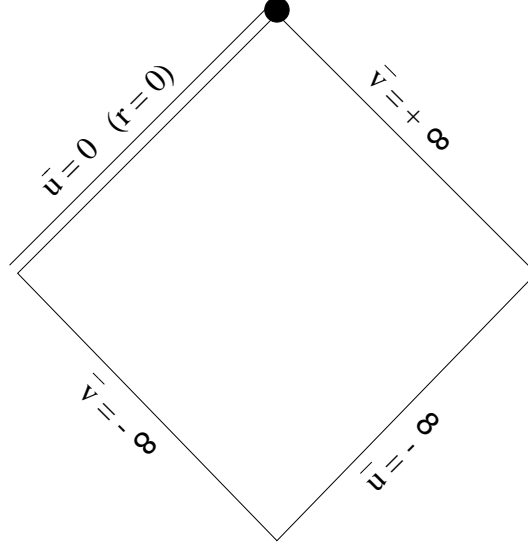

Figure 2: Causal structure for $q = n$ odd.

Figure 3: Penrose diagram of our new CSS solution (3.7). The null line $\bar{u} = 0$ is a singularity in the causal structure.
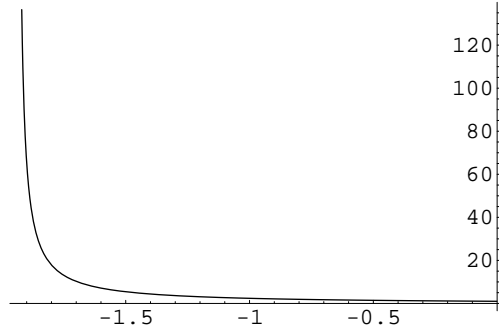

Figure 4: Numerical plot of the function $\rho(x)$ as derived from the system (4.25) with $\rho(0) = 0$ and $\rho'(0) = -2/3$, showing the divergence of ρ for $x \to x_1$ as the AdS boundary is approached (the behaviour is given in the first of Eqs. (4.24)).
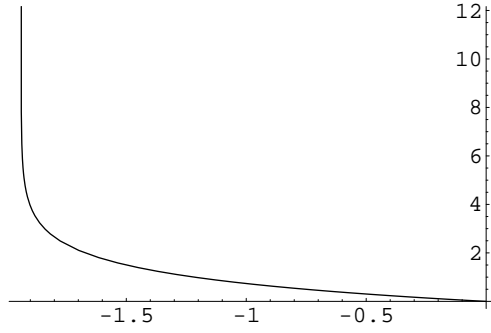
Figure 5: Numerical graph of $\sigma(x)$ starting from $\sigma(0) = 0$. In the limit $x \to x_1$ this is well represented in the second of Eqs. (4.24).

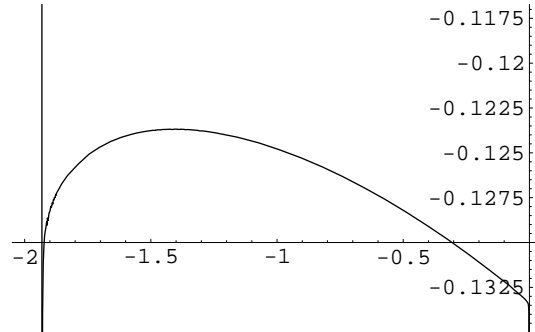Figure 6: Plot of $\psi'(x)$. In particular it is clear that $\psi''(x) \to \infty$ as $x \to x_1$. This feature is reproduced in the third of Eqs. (4.24) (giving $\psi'' \sim \ln(x - x_1)$).